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A variational approach to nonlinear two-point boundary value problems

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ABSTRACT

A variational formulation is established for a nonlinear two-point boundary value problem, an analytical solution is obtained using the Ritz method, and the obtained solution is valid for the whole solution domain.

1. Introduction

Two-point boundary value problems arise in applied mathematics, theoretical physics, engineering, control and optimization theory etc. In this paper, we consider two-point boundary value problems of the following type [1].

$$y'' = f(x, y, y'), \quad a \le x \le b \quad y(a) = 0, \quad y(b) = 0. \tag{1}$$

The two-point boundary value problems were studied by Wazwaz using the decomposition method [2]. The main demerit of the decomposition method is the difficulty in calculating the so-called Adomian polynomials [3]. In this paper, a variational approach [4–13] is applied to the discussed problem.

2. Implementation of the variational approach

In order to use the variational method [4–13], we have to establish a variational formulation for the discussed problem first. To illustrate the solution procedure, we consider the following examples.

Example 1. We first consider the following boundary value problem

$$Y'' + \frac{2}{x}Y' + Y^3 + 3xY^2 + 3Yx^2 + x^3 + \frac{2}{x} - 6 - x^6 = 0 \tag{2}$$

with the boundary conditions

$$Y(0) = Y(1) = 0. \tag{3}$$

The above equation, by a simple transformation, $y = Y + x$, turns out to be Example 2 in Ref. [2] where the decomposition method was used.

Applying the semi-inverse method [11–13], we obtain the following variational functional

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}x^2(Y')^2 + x^2\left(\frac{1}{4}Y^4 + xY^3 + \frac{3}{2}x^2Y^2 + \left(\frac{2}{x} - 6 + x^3 - x^6\right)Y\right)\right] dx. \tag{4}$$

Assume that the solution can be expressed in the following form

$$Y(x) = x(x-1)(ax+b) \tag{5}$$

where a and b represent unknown constants to be further determined.

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Submitting Eq. (5) into Eq. (4) and making $J[Y(x)]$ stationary with respect to a and b result in

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{371}{3960} + \frac{1}{15015}a^3 + \frac{1}{2145}ab^2 + \frac{3}{10010}a^2b - \frac{92}{1155}a \\ &\quad + \frac{1}{3960}b^3 - \frac{11}{120}b - \frac{3}{2860}a^2 - \frac{1}{330}ab - \frac{1}{440}b^2 = 0 \end{aligned} \tag{6}$$

$$\frac{\partial J}{\partial b} = \frac{1151}{9240} + \frac{1}{2145}a^2b + \frac{1}{10010}a^3 + \frac{1}{1320}ab^2 + \frac{1}{2310}b^3 - \frac{17}{140}b - \frac{11}{120}a - \frac{1}{280}b^2 - \frac{1}{660}a^2 - \frac{1}{220}ab = 0. \tag{7}$$

Solving Eqs. (6) and (7) simultaneously, we can determine the values of a and b: $a = 0$ and $b = 1$.

We, therefore, obtain

$$Y(x) = x^2 - x \tag{8}$$

which is the exact solution.

The solution process is simpler than that of the decomposition method [2].

Example 2. We consider the following equation

$$Y'' + \frac{4}{x}(Y' + 1) + (Y + x + 2)^2 - 4 - 18x - 4x^3 - x^6 = 0 \tag{9}$$

with the boundary conditions

$$Y(0) = Y(1) = 0. \tag{10}$$

Eq. (9) becomes Example 3 in Ref. [2] by a transformation, $y = Y + x + 2$.

The variational formulation can be easily established using the semi-inverse method [11–13]

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}x^4(Y')^2 + x^4\left(\frac{1}{3}(Y + x + 2)^3 + \left(\frac{4}{x} - 4 - 18x - 4x^3 - x^6\right)Y\right)\right] \mathrm{d}x. \tag{11}$$

Assume that the solution can be expressed as

$$Y(x) = x(x - 1)(ax + b) \tag{12}$$

where a and b are unknown constants to be further determined.

Proceeding in a similar way as before, we have

$$\frac{\partial J}{\partial a} = \frac{223}{1716} - \frac{19}{315}a - \frac{3}{44}b - \frac{1}{4004}a^2 - \frac{1}{1430}ab - \frac{1}{1980}b^2 = 0 \tag{13}$$

$$\frac{\partial J}{\partial b} = \frac{719}{4680} - \frac{1}{12}b - \frac{3}{44}a - \frac{1}{1320}b^2 - \frac{1}{2860}a^2 - \frac{1}{990}ab = 0. \tag{14}$$

Solving Eqs. (13) and (14) simultaneously yields a and b.

We, therefore, obtain

$$Y(x) = x^3 - x \tag{15}$$

which is the exact solution.

Example 3. As the last example, we consider the following boundary value problem [2]

$$y'' + \pi^3\frac{y^2}{\sin(\pi x)} = 0 \tag{16}$$

with the boundary conditions

$$y(0) = y(1) = 0. \tag{17}$$

Its variational formulation reads

$$J[Y(x)] = \int_0^1 \left[-\frac{1}{2}(Y')^2 + \frac{\pi^3 y^3}{3\sin(\pi x)}\right] \mathrm{d}x. \tag{18}$$

Assume that the solution has the following form

$$Y(x) = x(x - 1)(ax + b) \tag{19}$$

where a and b are unknown constants to be further determined.

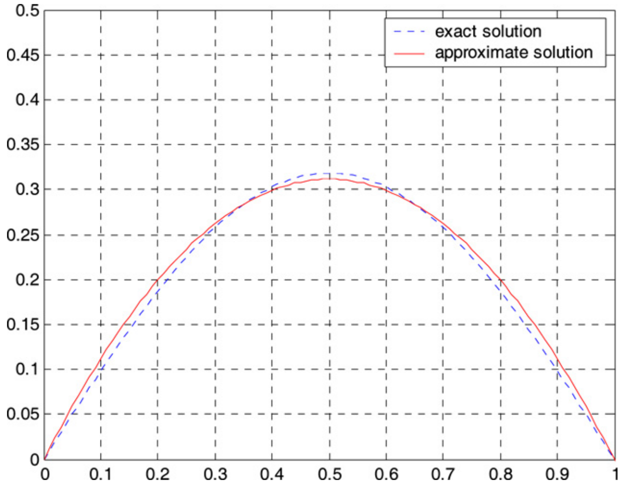

Fig. 1. Comparison of the approximate solution with the exact one. Continuous line: approximate solution; discontinuous line: exact solution.

By a similar manipulation as illustrated in previous examples, we have

$$\begin{aligned}\frac{\partial J}{\partial a} &= \frac{1}{16888498602639360}(126433853087102330656950a^2\varsigma(9)) - 2251799813685248a\pi^9\\&+ 730490997241668483 90b^2\pi^4\varsigma(5) + 243496999080556161300\pi^4\varsigma(5)\\&- 748164166529773366575b^2\pi^2\varsigma(7) - 1945226832977410753095 0ab\pi^2\varsigma(7)\\&+ 168578470782803107542600ab\varsigma(9) - 2814749767106560b\pi^9\\&- 1421511916406569396492 5a^2\pi^2\varsigma(7) + \frac{146098199448333696780a^2\varsigma(5)}{\pi^5} = 0 \end{aligned}\tag{20}$$

$$\begin{aligned}\frac{\partial J}{\partial b} &= \frac{1}{16888498602639360}(-5629499534213120b\pi^9 + 146098199448333696780\pi^4\varsigma(5)\\&+ 121748499540278080650a^2\pi^4\varsigma(5) - 1496328333059546733150ab\pi^2\varsigma(7)\\&- 972613416488705376547 5a^2\pi^2\varsigma(7) + 146098199448333696780b^2\pi^4\varsigma(5)\\&- 1496328333059546733150b^2\pi^2\varsigma(7) + 84289235391401553771300a^2\varsigma(9)\\&- 2814749767106560a\pi^9)/\pi^9 = 0.\end{aligned}\tag{21}$$

Eqs. (20) and (21) lead to the results $a = 0$ and $b = -1.2463$.

We, therefore, obtain

$$y(x) = -1.2463x^2 + 1.2463x. \tag{22}$$

Fig. 1 shows high accuracy of the obtained solution.

3. Conclusion

In this paper, we apply the variational approach to the nonlinear two-point boundary value problems. The obtained solutions show that the method is of remarkable simplicity, while the results are of high accuracy. The method can be easily extended to other boundary value problems.

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